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Brick walls for black holes in AdS/CFT

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Abstract

We study the 't Hooft's brick wall model for black holes in a holographic context. The brick wall model suggests that without an appropriate near horizon IR cut-off, the free energy of the probe fields shows the divergence due to the large degenerate states near the horizons. After studying the universal nature of the divergence in various holographic settings in various dimensions, we interpret the nature of the divergence in a holographic context. The free energy divergence is due to the large degeneracy and continuity of the low energy spectrum in the boundary theory at the deconfinement phase. These divergence and continuity should be removed by finite N effects, which make the spectrum discrete even at the deconfinement phase. On the other hand, in the bulk, these degenerate states are localized near the horizon, and the universal divergence of these degenerate states implies that the naive counting of the degrees of freedom in bulk should be modified once we take into account the non-perturbative quantum gravity effects near the horizon. Depending on the microscopic degrees of freedom, the position, where the effective field theory description to count the states breaks down, has different Planck scale dependence. It also implies the difficulty to have an electron like gauge-singlet elementary field in the boundary theory Lagrangian. These singlet fields are at most composite fields, because they show divergent free energy, suggesting a positive power of N at the deconfinement phase.

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1. Introduction

Understanding the quantum nature of gravity has been one of the most exciting topics in high energy physics, and black holes are touchstones of our understanding of the quantum nature of gravity. The recent developments of the AdS/CFT correspondence, or more broadly, the gauge/gravity duality [1–3], give many new important insights about black holes. Recent developments of various aspects of the quantum nature of black holes, including their microscopic entropy counting [4], the Hawking–Page transition [5], the quantum nature for information paradox including unitarity [6], are related to deepening our understanding of gauge/gravity duality. Moreover gauge/gravity duality defines quantum gravity non-perturbatively.

The brick wall model is the model for black holes which 't Hooft proposed [7]. He pointed out that the degrees of freedom near the black hole event horizon, evaluated through the probe field free energy or entropy, always diverge. This divergence is due to the infinite warped factor of the metric near the horizon. By requiring that diverging free energy/entropy be finite, we *have to* introduce the near horizon effective cut-off. The importance of brick wall model is the necessity of this cut-off. Furthermore, 't Hooft pointed out that by requiring it to be the same order of the background black hole's one, the cut-off scale ends up to be Planck scale measured by invariant distances. Given the non-perturbative quantum gravity from dual field theory, it is very natural to ask what the brick wall model implies in the gauge/gravity setting. The purpose of this paper is to understand this point.

In this paper we revisit these brick wall model results from the dual field theory viewpoint. First we study the universal nature of the brick wall model in various exotic black brane backgrounds for the probe fields. These analyses are done for the probe fields not only the scalar but also fermions, which are either charged or not, on the background which has non-trivial IR dynamics. Then, we interpret the Planck-scale cut-off dependence of the brick wall models as the large N dependence of the free energy of the probe fields. These analyses suggest us that in dual field theory, it is difficult to have a gauge-free singlet object unless it is a composite object.

In order to simplify the bulk argument, in this paper we take the limit where g_s is fixed finite and N to be very large, so $\lambda \equiv g_s N$ is also very large.² Note that this is different from 't Hooft limit ($g_s \rightarrow 0$, $N \rightarrow \infty$ with $\lambda \equiv g_s N$ fixed finite). Therefore, in our bulk analysis we take both string scale l_s and Planck scale l_p of the same order, $l_s \sim l_p$ and they are very small $l_s \sim l_p \ll 1$, where other macroscopic quantities including AdS scale are assumed to be $\mathcal{O}(1)$. This corresponds, in boundary, to the stronger coupling limit than the 't Hooft limit since $\lambda/N \sim g_{YM}^2$ fixed finite with λ very large as $\lambda \sim N$. Due to $g_s \sim 1$, in bulk very short scale distance such as string scale and Planck scale, it is different from usual perturbative string theory where $g_s \rightarrow 0$. However since we consider the limit $l_s \sim l_p \ll 1$, we have low energy effective theory description. In this limit, we will discuss when and where the classical gravity description breaks down. In our limit, both stringy effects and quantum gravity effects appear at the same level.

Another important assumption is that the whole gauge theory system becomes a thermal equilibrium. Thermal equilibrium for the system is assumed for the gravity side, which is also the basic assumption of the brick wall model. This implies that we will consider only the large black hole in the AdS space–time, where the curvature of the black hole is AdS scale.

² This limit has been considered, see for examples [8,9].

The organization of this paper is the following; In Section 2, we briefly review the known results about the original brick wall model for black holes by 't Hooft and see the probe field free energy show divergence due to the near horizon regime. In Section 3, we study the universal nature of the brick wall in various exotic black branes including the recently studied Lifshitz black branes [13–15], hyperscaling violating black branes [16,17]. We will see its universal nature that under the mild conditions, probe field free energy always diverge and its equality with the background exotic black brane yield the Planck scale. Part of the results of Section 3 is already known in the literature [18].

In Section 4, we interpret these IR divergences from the dual field theory viewpoint. We interpret this universal divergence of the probe field without brick wall as an indication of difficulty to introduce the gauge free (singlet) field in the dual boundary theories. This is the same statement that the $SU(N)$ gauge singlet fields, such as a fundamental electron field, are difficult to be introduced in holographic setting, unless they are the composite fields (like meson), which implies that they cannot be the fundamental fields in the Lagrangian level. There composite fermions show divergent free energy in the large N limit, which is reflected to the universal divergent free energy of the probe field near the horizon without the brick wall cut-off. The brick wall introduces the Planck scale dependent cut-off, and this is the same as keeping the N finite. This is one of the main result of this paper.

We also discuss Euclidean path integral measure and its apparent non-diffeomorphism invariance. However this result is already known in the literature [44–47].

Before we proceed, we comment on the connection on the brick wall to the entanglement entropy and also various related references. In [10,11], the entanglement entropy is shown to be equivalent to the thermal entropy in Euclid Rindler Hamiltonian, and based on this, the black hole entropy is interpreted as entanglement entropy associated with the 't Hooft's brick wall model in [10,11]. In this paper, however, we take a different interpretation. Based on the modern viewpoint, we assume that black hole entropy has dual microscopic origin from holographic viewpoint [4]. In this paper, we mainly consider the probe field added on top of above black holes and discuss mainly the implication of the probe field free energy divergence and its connection to dual field theory degrees of freedom.

2. The brick wall model by 't Hooft

2.1. Quantization for probe fields

In this section, we review the original 't Hooft's brick wall model [7]. First, we review the quantization of the probe scalar fields, and then we evaluate the partition function and free energy. Here we consider probe scalar fields, however, the following discussion will be applied to the other fields, for example the gravitons around the background metric, without essential modifications.

The situation we consider is space–time where we assume the homogeneity and isotropy in fixed r coordinate, where r is the radial coordinate. Under such mild assumption, we will conduct the analysis in the generic setting. Given this assumption, the metric is specified by $g_{tt}(r)$, $g_{rr}(r)$, $g_{\vec{x}\vec{x}}(r)$ only. \vec{x} represents the d dimensional spatial directions in $D = d + 2$ space–time. By using the r coordinate redefinition, we take the gauge that $-g_{tt}(r) = (g_{rr}(r))^{-1}$, so the most generic metric in this situation is always written as

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + g_{\vec{x}\vec{x}}(r)d\vec{x}^2. \quad (1)$$

Since we consider the black hole (black brane) geometry, we assume $g_{tt}(r) = 0$ at $r = r_h$, where r_h is the radial position of the horizon.

Consider, for simplicity, the minimally coupled scalar field, with mass m in this background. The scalar field wave equation is

$$(\square + m^2)\phi = 0. \quad (2)$$

We will consider the quantization of these fields by imposing two boundary conditions. Since we solve the second order differential equations, we have two independent solutions. One boundary condition forces us to take the appropriate linear combinations of the two solutions. Imposing the other boundary conditions, we obtain the quantization condition for the spectrum. Let us see this more concretely.

First, we impose the “UV cut-off” for the scalar field

$$\phi \rightarrow 0 \quad (r = L \rightarrow \infty). \quad (3)$$

If we impose Eq. (3) for the asymptotically Anti-de Sitter (AdS) setting, this is nothing but the requirement that the scalar field ϕ have no non-normalizable mode in the asymptotic AdS region for the positive mass scalar, $m \geq 0$ case. Remember that asymptotic AdS_D, the scalar field behaves

$$\phi \sim r^{\Delta_{\pm}}, \quad \Delta_{\pm} = D/2 \pm \sqrt{(D/2)^2 + m^2}. \quad (4)$$

Instead, if we impose Eq. (3) in the asymptotic flat case, this is the same as imposing the Dirichlet boundary condition at the spatial infinity, or normalizable condition for the fields.

Second, we impose the “IR cut-off” for the scalar field as ’t Hooft [7],

$$\phi = 0 \quad (r = r_h + h), \quad (5)$$

where $r = r_h$ is the horizon of the black hole and $h > 0$ is some very small distance scale which we will determine later. This imposes the Dirichlet boundary condition for the scalar field near the horizon. Actually we will see later that the Dirichlet boundary condition is not the crucial, we can impose either Dirichlet boundary condition with any constant values C_{r_h+h}

$$\phi = C_{r_h+h} \quad (r = r_h + h), \quad (6)$$

or instead Neumann boundary condition at $r = r_h + h$. For a moment, let’s first consider the Dirichlet boundary condition, Eq. (5).

Given the two boundary conditions (3) and (5) for the second order differential equation, the allowed mode is always quantized, and its spectrum E is parameterized by discrete parameters (m, \vec{k}) and integer n .

Let us see this more concretely. Taking the ansatz,

$$\phi = \phi(r)e^{-iEt+i\vec{k}\vec{x}} \quad (7)$$

the wave equations for $\phi(r)$,

$$\frac{1}{\sqrt{-g}}\partial_r(\sqrt{-g}g^{rr}\partial_r\phi) - E^2g^{tt}\phi - \vec{k}^2g^{\vec{x}\vec{x}}\phi - m^2\phi = 0, \quad (8)$$

allow generic solutions

$$\phi(r, E, \vec{k}, m) = c_1(E, \vec{k}, m)F^{(1)}(r, E, \vec{k}, m) + c_2(E, \vec{k}, m)F^{(2)}(r, E, \vec{k}, m), \quad (9)$$

where $F^{(1)}(r, E, \vec{k}, m)$ and $F^{(2)}(r, E, \vec{k}, m)$ are two independent solutions for Eq. (8) and $c_1(E, \vec{k}, m)$ and $c_2(E, \vec{k}, m)$ are constant w.r.t. r coordinate.

We have freedom to re-define any linear combinations of $F^{(1)}$ and $F^{(2)}$ as new $F^{(1)}$ and $F^{(2)}$, so by using this freedom, we can always take the choice such that $F^{(1)}$ satisfy the UV boundary condition (3). Generically for this choice of $F^{(1)}$, $F^{(2)}$ do not satisfy the boundary condition (3), therefore it forces us to set

$$c_2(E, \vec{k}, m) = 0. \quad (10)$$

Given this, the 't Hooft's IR boundary condition (5) gives the quantization condition for E , by

$$F^{(1)}(r = r_h + h, E, \vec{k}, m) = 0. \quad (11)$$

This condition yields discreteness for the energy eigenvalues E . Here we have assumed that we have also the IR cut-off along the \vec{x} directions therefore, \vec{k} is also quantized. We label the discrete energy eigenvalues satisfying condition (11) as E_n , where n is positive integer and we take $E_n < E'_n$ for $n < n'$.

2.2. Partition function for probe fields

Given the discrete spectrum for the scalar field around black hole background, we will consider the canonical ensemble for this scalar field with temperature given by the Hawking temperature of the black hole $1/\beta$.

For that purpose, it is convenient to introduce the occupation number $g(E)$ below the energy E . First, we take the ansatz

$$\phi = \phi_0 e^{-iEt + i \int k_r(r) dr + i \vec{k} \vec{x}}, \quad (12)$$

where ϕ_0 is constant, which is the same as setting

$$\phi(r) = \phi_0 e^{i \int k_r(r) dr} \quad (13)$$

to define $k_r(r)$ for $\phi(r)$ in Eq. (7). Then we have quantization conditions from Eq. (11)

$$n\pi = \int_{r_h+h}^L k_r(r, E_n, \vec{k}) dr \quad (\text{for positive integer } n), \quad (14)$$

with $L \rightarrow \infty$. This gives the discrete spectrum labeled by n, \vec{k}, h, m as

$$E = E_n(\vec{k}, h, m). \quad (15)$$

Therefore, by integrating over the \vec{k} modes, the occupation number $g(E)$ is given by

$$g(E) = \frac{V}{\pi} \int k_r(r, \vec{k}, E) dr d\vec{k} \quad (16)$$

where V is proportional to the volume of the field theory spatial dimensions defined by

$$V \equiv \frac{1}{(2\pi)^d} \int d\vec{x}, \quad (17)$$

and it excludes r direction.³ Here we have also approximated that the volume is large enough so that the mode summation is given by $\sim V d\vec{k}$.

Then we find that the partition function for the scalar field ϕ is given by

$$\begin{aligned} Z_\phi &= \prod_{\vec{k}} \prod_n \sum_{m=0}^{\infty} e^{-\beta m E_n} \\ &= \prod_{\vec{k}} \prod_n \frac{1}{1 - e^{-\beta E_n}}, \end{aligned} \quad (18)$$

where n is for discrete spectrum, \vec{k} in $\prod_{\vec{k}}$ is taken for the integer quantum numbers and the second summation over m is due to the boson. Therefore,

$$\begin{aligned} \beta F_\phi &= V \int d\vec{k} \sum_n \log(1 - e^{-\beta E_n}) \\ &= \int dE \left(\frac{\partial g(E)}{\partial E} \right) \log(1 - e^{-\beta E}) \\ &= - \int dE \frac{\beta g(E)}{(1 - e^{-\beta E})} \\ &= - \frac{\beta V}{\pi} \int \frac{dE}{(1 - e^{-\beta E})} dr \left(d\vec{k} k_r(r, \vec{k}, E) \right), \end{aligned} \quad (19)$$

where \vec{k} is approximate as the continuous variables. Given the wave equations for $k_r(r, \vec{k}, E)$, we can evaluate this. There is trivial volume dependence (19) in the above expression, which we will omit since we will always consider free energy/entropy per unit volume.

We have considered the specific Dirichlet boundary condition for both UV and IR boundary conditions in Eqs. (3) and (5). For the free energy evaluation, the effects of the boundary condition only affect the quantization conditions (14), (15), (16). If we consider more generic Dirichlet boundary condition, where in UV, we choose $\phi = C_\infty$, and in IR, we choose $\phi = C_{r_h+h}$, then, quantization condition is shifted as

$$\begin{aligned} C_\infty &= e^{i \int_{r_h+h}^L dr k_r(r) + 2\pi i n} C_{r_h+h} \\ \Rightarrow -i \log\left(\frac{C_\infty}{C_{r_h+h}}\right) &= \int_{r_h+h}^L dr k_r(r) + 2\pi n. \end{aligned} \quad (20)$$

Eq. (14) corresponds to $C_\infty/C_{r_h+h} = e^{i\pi n}$ with $C_{r_h+h} \rightarrow 0$. However it is clear that these modifications of the boundary conditions will not qualitatively change the $g(E)$ for Eq. (16). Furthermore, if we replace both UV and IR boundary condition by Neumann boundary condition, $g(E)$ is not modified qualitatively in that case too. In this way, for the free energy, these modifications of the boundary condition will not affect the argument below qualitatively.

Using the WKB approximation, from Eqs. (8) and (13), we obtain

$$k_r(r) = \sqrt{g_{rr}(r) \left(-E^2 g^{tt}(r) - \vec{k}^2 g^{\vec{x}\vec{x}}(r) - m^2 \right)}. \quad (21)$$

³ In holographic setting r direction does not appear in the dual field theory.

Therefore, the integration $d\vec{k}$ in $\int d\vec{k} k_r(r)$ can be conducted, and written as

$$\begin{aligned} \int d\vec{k} k_r(r) &= \frac{1}{2} \gamma(S^{d-1}) \int d|\vec{k}|^2 |\vec{k}|^{d-2} k_r(r) \\ &= \frac{c_d}{2} \gamma(S^{d-1}) \left(\frac{E^2 - m^2 f(r)}{f(r)^2} \right)^{\frac{d+1}{2}} (\rho f(r))^{\frac{d}{2}}, \end{aligned} \quad (22)$$

with the factor

$$c_d = \int_0^1 dy y^{\frac{d-2}{2}} \sqrt{1-y} = \frac{\sqrt{\pi} \Gamma\left(2 - \frac{2}{d}\right)}{2\Gamma\left(\frac{7}{2} - \frac{2}{d}\right)}, \quad (23)$$

in general dimensions. Here $\gamma(S^{d-1})$ is area of the unit radius S^{d-1} , $f(r) \equiv -g_{tt}(r) = g^{rr}(r)$ and $\rho \equiv g_{\vec{x}\vec{x}}(r)$. The WKB approximation to obtain Eq. (21) is justified if

$$\frac{(\sqrt{-g} g^{rr} k_r(r))'}{\sqrt{-g} g^{rr} k_r^2(r)} \ll 1, \quad (24)$$

is satisfied. In the near horizon region, which we are most interested in, $g^{\vec{x}\vec{x}}$ approach constant, but g^{tt} diverges, therefore we are in the range $-E^2 g^{tt}(r) \gg \vec{k}^2 g^{\vec{x}\vec{x}}(r)$, $-E^2 g^{tt}(r) \gg m^2$, and we can approximate as,

$$k_r(r) \approx \sqrt{-g_{rr}(r) g^{tt}(r)} E = g_{rr}(r) E \quad (25)$$

in the gauge $g_{rr}(r) = -g^{tt}(r)$. Then we see that $\sqrt{-g} g^{rr} k_r(r)$ approach constant value, therefore WKB approximation (24) is satisfied.⁴

So far the argument is for generic dimensions. The integration over \vec{k} along (22) simplifies for the $d = 2$ case. In that case, it simplifies as

$$\begin{aligned} \int d\vec{k} k_r(r) &\sim -\frac{1}{g_{rr}(r) g^{\vec{x}\vec{x}}(r)} (k_r(r))^{3/2} \Big|_{|\vec{k}|=0}^{k_r(r)=0} \\ &= \frac{1}{g_{rr}(r) g^{\vec{x}\vec{x}}(r)} \left(g_{rr}(r) \left(-E^2 g^{tt}(r) - m^2 \right) \right)^{3/2} \\ &\sim \frac{(-g_{rr}(r) g_{tt}^{-1}(r))^{3/2}}{g_{rr}(r) g^{\vec{x}\vec{x}}(r)} E^3 \\ &\sim (-g_{tt})^{-3/2} g_{rr}(r)^{1/2} g^{\vec{x}\vec{x}} E^3, \end{aligned} \quad (26)$$

where we have neglected the effect of mass term and also irrelevant numerical factors. The mass term is suppressed near the horizon compared with E since

$$-E^2 g^{tt}(r) \gg m^2 \quad (27)$$

near the horizon, from which the dominant contributions come. Given $\int d\vec{k} k_r$, using (19), we can obtain the free energy of the probe. What 't Hooft pointed out in [7] is that this contribution diverges due to the near horizon contribution.

⁴ Later we will also consider large momentum region $-E^2 g^{tt}(r) \gtrsim -\vec{k}^2 g^{\vec{x}\vec{x}}(r)$, there, the WKB approximation is not strictly valid. However, we neglect such subtle issues in this paper since we expect that the result will not change much qualitatively.

2.3. Schwarzschild black holes in asymptotic flat space–time in 4 dimensions

Let us review the 't Hooft's original asymptotic flat Schwarzschild black hole case in $D = 4$ dimensions. In this subsection, we take the 4d Planck scale set to be unit. Note that if we consider the black holes, then due to the centrifugal force, \vec{k} integration is replaced by the l integration and we have additional r^2 factor and then, using that $g_{rr}(r)$ has single zero at the horizon in Schwarzschild black holes and

$$\begin{aligned} g_{rr}^2|_{r \approx r_h} &\approx (1 - \frac{r_h}{r})^{-2}, \quad -g_{tt}^{-1}(r) = g_{rr}(r), \\ g_{ii}|_{r=r_h+h} &= r_h^2 + O(h), \end{aligned} \quad (28)$$

so Eq. (26) gives the dominant contributions

$$\int dr dl k_r(r) \sim \int dr r^2 g_{rr}(r)^2 E^3 \sim E^3 (\frac{L^3}{3} + \frac{r_h^4}{h}), \quad (29)$$

at $L \rightarrow \infty$ and $h \rightarrow 0$ limit. The necessity of factor r^2 is clear from the dimensional analysis, $d\vec{k} = dl(2l+1)/r^2$ in $D=(d+2)=4$ case.

Therefore, using (19) in the black hole case as 't Hooft, after E integration, we obtain the free energy per unit volume,

$$F_\phi \sim -\frac{1}{h} (\frac{r_h}{\beta})^4 - L^3 \int_m^\infty dE \frac{(E^2 - m^2)^{3/2}}{e^{\beta E} - 1}. \quad (30)$$

This is the free energy of the probe scalar field around the black hole, where it is thermal equilibrium with the black hole. This free energy diverges at $h \rightarrow 0$. However it does not make sense that the probe free energy diverges, and gets bigger than the background black brane free energy.⁵ This forces us to put the cut-off for the minimal values for h . The black hole entropy is

$$S_{BH} = \frac{r_h^2}{l_p^2} \Rightarrow F_{BH} \sim \frac{S_{BH}}{\beta} = \frac{r_h}{l_p^2} \quad (31)$$

where l_p is 4-dimensional Planck scale. The Hawking temperature T is

$$\beta = \frac{1}{T} = r_h. \quad (32)$$

A natural cut-off is given by the background black hole free energy, this gives the cut-off for h as

$$F_\phi \lesssim F_{BH} \Rightarrow h \lesssim \frac{l_p^2}{r_h} \sim T l_p^2 \quad (33)$$

Actually this distance h measured in the coordinate invariant way gives

⁵ In this case, since it is asymptotic flat space–time, we do not have clear holographic interpretation. Later we discuss in more detail about the probe free energy and background free energy contribution from gauge/gravity viewpoint in asymptotic AdS.

$$\begin{aligned}\Delta s &= \int_{r_h}^{r_h+h} dr \sqrt{g_{rr}} \\ &\sim \int_{r_h}^{r_h+h} \sqrt{r_h} \frac{dr}{\sqrt{r-r_h}} = \sqrt{r_h h} = l_p\end{aligned}\quad (34)$$

so it is $O(1)$ in Planck unit. Especially this means that it is *independent* of the black hole states (mass, temperature) we consider for the background, and implies rather it is some intrinsic nature of the theory.

2.4. Origin of the divergence

The essential parts of the 't Hooft's brick wall model is the necessity of the near horizon cut-off at $r = r_h + h$, with $h > 0$. The origin of the free energy divergence is due to the divergence of the occupation number $g(E)$ given in Eq. (16) and this is associated with the infinite volume of the deep throat region near the horizon. Note that $g(E)$ diverges at finite E means that there are infinitely *low energy* spectrum degeneracy.

As the spatial volume gets bigger in the system, the allowed low energy excitations increase. Therefore, given the temperature, we have more entropy. This is physically due to the large IR regime near the horizon, we can have infinitely low energy quantized mode allowed. This is exactly happening in the near horizon. However, considering the infinite volume factor from the radial direction only, we have only a logarithmic divergence in

$$\int dr k_r \sim \int dr g_{rr} E \sim \frac{E}{T} \log h, \quad (35)$$

which is not what we have observed. Actually, in the near horizon region the volume of the \vec{k} direction also becomes bigger and then this divergence is enhanced after the \vec{k} integration as

$$\int dr d\vec{k} k_r(r) \sim \int dr g_{rr}^2 E^3 \sim \frac{E^3}{T^2 h} \quad (36)$$

in the $g_{tt}(r) = g^{rr}(r)$ gauge. This is the origin of the $\int dr d\vec{k} k_r \propto 1/h$ divergence.

In summary, k_r in Eq. (25) has g_{rr} , but the \vec{k} integration, also add another factor g_{rr} . The \vec{k} integration yields another factor g_{rr} because of the integration region

$$\begin{aligned}0 \leq |\vec{k}|^2 &\leq g_{\vec{x}\vec{x}}|_{r=r_h+h} \left(-E^2 g^{tt}|_{r=r_h+h} - m^2 \right) \\ &\sim E^2 g_{\vec{x}\vec{x}} g_{rr}|_{r=r_h+h} \sim \frac{T}{h}\end{aligned}\quad (37)$$

near the horizon. Therefore if we introduce the boundary field theory momentum cut-off Λ ,

$$|\vec{k}|^2 \leq \Lambda^2 \ll E^2 g_{\vec{x}\vec{x}} g_{rr}|_{r=r_h+h} \sim \frac{T}{h} \quad (38)$$

Then, the free energy has divergence only by $F \sim \Lambda^2 \log h$, since

$$d\vec{k} k_r(r) \sim \int \Lambda^2 g_{rr} E \quad (39)$$

instead of Eq. (26), and this gives

$$F \sim \Lambda^2 T \log h \quad (40)$$

using Eq. (19). It depends on the theory whether the theory admits momentum cut-off Λ satisfying (38) or not.

Later we discuss this point from the generic holographic viewpoint. There, since it approaches the asymptotic AdS, which is scale invariant theory, it is unnatural to have such a momentum cut-off scale Λ from the boundary viewpoint.

It is pointed out in [12] that a Pauli–Villars regulator induces the similar finite IR cut-off effect. In [12], a Pauli–Villars regulator field $\phi^{(PV)}$ is introduced, where $\phi^{(PV)}$ has opposite sign to the ϕ field and has large mass M with $m^2 \ll M^2$. Under the same WKB approximation, the regulator field $\phi^{PV}(r) = \phi^{PV}(r)_0 e^{i \int k_r^{(PV)}(r) dr}$ shows the relation which is very similar to (21) but m^2 is replaced by M^2 as

$$k_r^{(PV)}(r) = \sqrt{g_{rr}(r) \left(-E^2 g^{tt}(r) - \vec{k}^2 g^{\bar{x}\bar{x}}(r) - M^2 \right)}. \quad (41)$$

This $k^{(PV)}$ approaches k_r for ϕ given by (21) at

$$-E^2 g^{tt}(r) \gg M^2, \quad \vec{k}^2 g^{\bar{x}\bar{x}}(r) \gg M^2. \quad (42)$$

These are equivalent to

$$r - r_h \ll \frac{T}{M^2}, \quad \vec{k}^2 \gg M^2. \quad (43)$$

However as we have seen in this subsection, the divergence of the free energy is exactly originated by the many modes in this parameter region. Therefore a Pauli–Villars regulator $\phi^{(PV)}$ with mass M plays the same role as introducing the cut-off h with

$$h \sim \frac{T}{M^2}. \quad (44)$$

Since the local temperature at $r = r_h + h$ is

$$T_{local} = \frac{T}{\sqrt{g_{tt}|_{r=r_h+h}}} \sim M, \quad (45)$$

this regulator $\phi^{(PV)}$ removes the degrees of freedom

$$E \gtrsim M, \quad \vec{k}_{local}^2 \gtrsim M^2, \quad (46)$$

where we define $\vec{k}_{local}^2 = g^{\bar{x}\bar{x}}|_{r=r_h+h} \vec{k}^2$, so that the divergent contribution is cut-off. In terms of h , this is

$$E \gtrsim \sqrt{\frac{T}{h}}, \quad \vec{k}_{local}^2 \gtrsim \frac{T}{h}. \quad (47)$$

It is clear that any regulator, other than above Pauli–Villars which removes the local modes (46), plays the same role as introducing nonzero cut-off h to remove the modes (47).⁶ And if M is Planck scale, the regulated free energy F_ϕ becomes the same order to the Gibbons–Hawking

⁶ Such a removal of UV degrees of freedom should be considered as a result of the strong gravitational interactions near the horizon. Note that this is a UV cut-off in the bulk gravitational theory, which corresponds to the IR cut-off in the holographic dual boundary theory through the UV/IR relation [53].

black hole free energy. Regarding the black hole entropy as this F_ϕ , and replacing the black hole by the brick wall is the original idea of 't Hooft's brick wall model for the black holes [7]. However in this paper, we interpret this divergence rather as an evidence of the composite nature of the any probe fields in holographic dual field theory. Since in dual field theory, we can have clear interpretation of the probe scalar fields as new degrees of freedom, it is more natural not to regard F_ϕ as the background black hole entropy F_{BB} . But for a moment, we keep this issues aside and continue the universal nature of the model. Then later in Section 4, we discuss dual interpretations.

3. Universality of brick wall models

In previous section, we have reviewed the original 't Hooft's brick wall model. In this section, we first consider the brick wall model for the 4-dimensional Schwarzschild black brane in asymptotic anti-de Sitter and see the same results hold. In some sense, this is expected since the brick wall model is determined by the near horizon nature and both Schwarzschild black brane in asymptotic flat space-time and in asymptotic AdS has the same near horizon.

Then, we will study more generic setting of the brick wall properties. We will now generalize the previously studied nature of the universality of the brick wall in more generic black brane and see how universal it is.

In order to check this universality, we consider more generic situation where the background black brane shows various exotic natures, some of which include the recently studied Lifshitz black branes [13–15], hyperscaling violating black branes [16,17]. There, the background scalar/gauge fields do not take a constant VEV but rather takes non-trivial dynamics. We will conduct the analysis for both neutral scalar and also the charged scalar on non-trivial background, with and without minimal coupling assumption. We will see its universality under the mild conditions, probe field free energy always diverge and its equality with the background exotic black brane yield the Planck scale. We will also check for the model on the probe brane degrees of freedom. Then we check similar things this for fermions.

3.1. Schwarzschild black brane in AdS₄

Let's first consider the Schwarzschild black brane in AdS₄. In this case, the metric is

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\vec{x}^2, \\ f(r) = r^2 - r_h^2. \quad (48)$$

Here we have set the AdS curvature scale to be unit.

Since $g_{rr}(r)$ has single zero at the horizon and we have

$$g_{rr}^2|_{r \approx r_h} \approx r_h^{-2}(r - r_h)^{-2}, \quad (49)$$

$$-g_{tt}^{-1}(r) = g_{rr}(r), \quad g_{\vec{x}\vec{x}}|_{r=r_h+h} = r_h^2 + O(h). \quad (50)$$

Therefore after r integration using Eq. (19), we have free energy divergence as $1/h$,

$$\int dr d\vec{k} k_r(r) \sim \int dr g_{rr}(r)^2 g_{\vec{x}\vec{x}}(r_h) E^3 \\ \sim E^3 \left(\frac{1}{L} + \frac{1}{h} \right) \quad (51)$$

where we have used $g_{rr}(r)^2 \rightarrow 1/r^4$, $g_{\bar{x}\bar{x}}(r) \rightarrow r^2$ at $r \rightarrow L \sim \infty$. At the $h \rightarrow 0$ limit, this free energy for the probe scalar field in black brane background diverges. And we obtain the free energy per unit volume

$$F_\phi = -\frac{1}{h} \frac{1}{\beta^4} - \frac{1}{L} \int_m^\infty dE \frac{E^3}{e^{\beta E} - 1}. \quad (52)$$

In the black brane case, the background black brane entropy is proportional to the spatial dimension volume, so the free energy per unit volume is

$$S_{BB} = \frac{1}{l_p^2} g_{\bar{x}\bar{x}}|_{r=r_h} \Rightarrow F_{BB} \sim \frac{S_{BB}}{\beta} = \frac{1}{l_p^2} \frac{r_h^2}{\beta} \quad (53)$$

Therefore, we have cut-off

$$h \sim \frac{l_p^2}{r_h^2 \beta^3}. \quad (54)$$

Since we have temperature in terms of r_h as

$$\beta = \frac{1}{r_h},^7 \quad (55)$$

therefore the cut-off h is

$$h \sim r_h l_p^2 = T l_p^2. \quad (56)$$

The distance h measured in the coordinate invariant way (the invariant distance)

$$\begin{aligned} ds &= \int_{r_h}^{r_h+h} dr \sqrt{g_{rr}} \\ &\sim \int_{r_h}^{r_h+h} \frac{1}{\sqrt{r_h}} \frac{dr}{\sqrt{r-r_h}} = \sqrt{\frac{h}{r_h}} \sim l_p. \end{aligned} \quad (57)$$

3.2. Lifshitz and hyperscaling violating black brane in 4 dimensions

3.2.1. Background near horizon solution

To study more generic setting, let's consider the following dilaton Maxwell and gravity model [16,17].

$$S = \int d^4x \sqrt{-g} \left\{ R - 2(\nabla\phi)^2 - e^{2\alpha\phi} F_{\mu\nu} F^{\mu\nu} - V_0 e^{2\delta\phi} \right\}. \quad (58)$$

This theory allows more exotic black brane solutions. We consider the case $V_0 < 0$. We are assuming that this is the effective theory valid near the horizon, which smoothly connect to the

⁷ To see this, note that our metric has in Euclidean signature, $ds^2 = r_h(r-r_h)d\tau^2 + \frac{dr^2}{r_h(r-r)}$. By coordinate changes $\tilde{r} = \int dr/\sqrt{r_h(r-r_h)} \sim \sqrt{(r-r_h)/r_h}$, we have $ds^2 = \tilde{r}^2 r_h^2 d\tau^2 + d\tilde{r}^2$. This gives $\beta \sim 1/r_h$, or $T \sim r_h$.

UV region where we have constant dilaton and the metric approaches AdS.⁸ Taking the metric ansatz,

$$ds^2 = -a(r)^2 dt^2 + \frac{dr^2}{a(r)^2} + b(r)^2(dx^2 + dy^2), \quad (59)$$

the Maxwell equations are satisfied by

$$F = \frac{Q_e}{f(\phi)b^2} dt \wedge dr, \quad (60)$$

and the remaining equations of motion can be conveniently expressed in terms of an effective potential [29]

$$V_{\text{eff}} = \frac{1}{b^2} \left(e^{-2\alpha\phi} Q_e^2 \right) + \frac{b^2 V_0}{2} e^{2\delta\phi}, \quad (61)$$

as

$$\begin{aligned} (a^2 b^2 \phi')' &= \frac{1}{2} \partial_\phi V_{\text{eff}}, & (a^2 b^2)'' &= -2V_0 e^{2\delta\phi} b^2, \\ \frac{b''}{b} &= -\phi'^2, & a^2 b'^2 + \frac{1}{2} a'^2 b^2 &= a^2 b^2 \phi'^2 - V_{\text{eff}}. \end{aligned} \quad (62)$$

Solving equations of motions with the ansatz,

$$a = C_a r^\gamma, \quad b = r^\beta, \quad \phi = k \log r, \quad (63)$$

we obtain

$$\begin{aligned} \beta &= \frac{(\alpha + \delta)^2}{4 + (\alpha + \delta)^2}, & \gamma &= 1 - \frac{2\delta(\alpha + \delta)}{4 + (\alpha + \delta)^2}, \\ k &= -\frac{2(\alpha + \delta)}{4 + (\alpha + \delta)^2}, & Q_e^2 &= -V_0 \frac{2 - \delta(\alpha + \delta)}{2(2 + \alpha(\alpha + \delta))}, \\ C_a^2 &= -V_0 \frac{(4 + (\alpha + \delta)^2)^2}{2(2 + \alpha(\alpha + \delta))(4 + (3\alpha - \delta)(\alpha + \delta))}. \end{aligned} \quad (64)$$

Finite temperature solution is obtained from the fact that the equations of motion continue to hold under the following shift

$$a^2 b^2 \rightarrow a^2 b^2 + Cr, \quad b^2 = \text{fixed}, \quad \phi = \text{fixed} \quad (65)$$

where C is any constant, which we will set as $C = C_a^2 r_h^{2\beta+2\gamma-1}$, for convenience. This allows us to obtain the finite temperature solution

$$a^2 = C_a^2 r^{2\gamma} \left(1 - \left(\frac{r_h}{r} \right)^{2\beta+2\gamma-1} \right) \quad (66)$$

In the near horizon, this becomes

$$a^2(r) \simeq (2\beta + 2\gamma - 1) (r - r_h) r_h^{2\gamma-1} C_a^2. \quad (67)$$

The temperature of the black brane solution is given by

$$T \sim C_a^2 r_h^{2\gamma-1}. \quad (68)$$

⁸ This can be achieved by setting the potential as $V(\phi) = 2V_0 \cosh 2\delta\phi$, for example, and $\phi \rightarrow 0$ in UV.

3.2.2. Scalar fields on this background

Let's consider the minimally coupled scalar field in this set-up. In a quite analogous way to Section 2, we obtain the wave equations using the WKB approximation. The argument of Sections 2.1, 2.2 till Eq. (26) still holds as far as $d = 2$. Since the metric is different we have,

$$g_{rr} = \frac{1}{a(r)^2} \quad (69)$$

and

$$\begin{aligned} \int dr d\vec{k} k_r(r) &\sim \int dr \frac{b(r)^2}{a(r)^4} E^3 \\ &\sim \frac{r_h^{2\beta} E^3}{C_a^4 r_h^{4\gamma-2} h} + (L \text{ dependent part}) \end{aligned} \quad (70)$$

where for the first relation, we used (26). So the free energy of the scalar F_ϕ per unit volume is

$$\begin{aligned} F_\phi &\sim \int \frac{dE}{1 - e^{-\beta E}} dr d\vec{k} k_r(r) \\ &\sim \frac{r_h^{2\beta}}{C_a^4 r_h^{4\gamma-2} h \beta^4} + (L \text{ dependent part}). \end{aligned} \quad (71)$$

On the other hand, the generalized Lifshitz black brane temperature is $T \sim C_a^2 r_h^{2\gamma-1}$, and the entropy of the black brane is

$$S_{BB} \sim \frac{1}{l_p^2} r_h^{2\beta} \quad (72)$$

so the free energy is

$$F_{BB} \sim T S_{BB} \sim \frac{1}{l_p^2} C_a^2 r_h^{2\gamma+2\beta-1}. \quad (73)$$

Comparing this with F_ϕ , we have cut-off of h as

$$F_{BB} \lesssim F_\phi, \quad (74)$$

$$\Rightarrow h \lesssim \frac{l_p^2}{\beta^4 C_a^6 r_h^{6\gamma-3}} \sim C_a^2 r_h^{2\gamma-1} l_p^2 \sim T l_p^2. \quad (75)$$

Then the invariant distance is

$$\Delta s = \int_{r_h}^{r_h+h} dr \sqrt{g_{rr}(r)} \sim \frac{\sqrt{h}}{C_a r_h^{(2\gamma-1)/2}} \sim l_p. \quad (76)$$

Again, it is Planck length in terms of invariant distance, and independent of β and γ which characterize IR.

3.3. Generic dimension argument

In this subsection, we will show that the universality we have seen is from the Rindler like geometry. Especially, we will extend the previous results to the Schwarzschild black brane in AdS_D space-time. In this subsection, we keep the all length scale explicit. However we neglect irrelevant numerical coefficients.

The geometry we consider here is the Rindler like geometry

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + \rho d\vec{x}^2, \\ f(r) = T(r - r_h), \quad (77)$$

for the near horizon, where t_H , r_h , $\rho = \rho(r_h)$ are constant and \vec{x} represents the $d = D - 2$ dimensional flat space with compactification such that $\int d\vec{x} = V$. T is the Hawking temperature. This geometry is the near horizon geometry of generic black branes.⁹ Then again, using the WKB approximation, we have

$$\int d\vec{k} k_r(r) = \frac{1}{2} \gamma(S^{d-1}) \int d|\vec{k}|^2 |\vec{k}|^{d-2} k_r(r) \\ = \frac{c_d}{2} \gamma(S^{d-1}) \left(\frac{E^2 + m^2 f(r)}{f(r)^2} \right)^{\frac{d+1}{2}} (\rho f(r))^{\frac{d}{2}}. \quad (78)$$

The factor c_d is given in Eq. (23), and it is order one irrelevant numerical factor, so we have

$$F_\phi \sim V \int dr \int \frac{dE}{1 - e^{-E/T}} \left(\frac{E^2}{f(r)^2} \right)^{\frac{d+1}{2}} (\rho f(r))^{\frac{d}{2}} \\ \sim V \rho^{\frac{d}{2}} (T)^{d+2} \int_{r_h+h}^{\infty} dr (f(r))^{-\frac{d+2}{2}} \\ \sim V \rho^{\frac{d}{2}} (T)^{\frac{d+2}{2}} h^{-\frac{d}{2}}, \quad (79)$$

again we have neglected the numerical factor and m^2 term which is not the leading contribution near the horizon since $f(r) \rightarrow 0$ at the horizon.

Note that since the mass term is in the end negligible due to the warped factor implies that the argument above is not unchanged for non-minimally coupled scalar fields. This is because the effects of non-minimal coupling, like curvature couplings, are included to “renormalize” the mass term. However in the end the mass term is not dominant compared with the other terms, so above argument hold more generic situations.

On the other hand, the BH entropy of this geometry is $S_{BB} \sim V \rho^{\frac{d}{2}} / (l_p)^d$. From this, we have

$$F_{BB} \sim T V \rho^{\frac{d}{2}} \frac{1}{(l_p)^d}. \quad (80)$$

Assuming $F_\phi \sim F_{BB}$, we have

⁹ For example, for the Schwarzschild black brane in AdS_D space-time, we have $T = r_h / (R_{AdS})^2$ as the Hawking temperature and $\rho = (r_h / R_{AdS})^2$. In more generic Lifshitz or hyperscaling violating geometry, this power is more exotic as is given in Eq. (68).

$$h \sim T l_p^2, \quad (81)$$

and the distance h measured in the coordinate invariant way

$$\begin{aligned} ds &= \int_{r_h}^{r_h+h} dr \sqrt{g_{rr}} \\ &\sim \int_{r_h}^{r_h+h} \frac{dr}{\sqrt{(r-r_h)T}} \sim \sqrt{\frac{h}{T}} \sim l_p, \end{aligned} \quad (82)$$

which is universal as before.

We find the local temperature at $r = r_h + h$ is

$$T_{local} = \frac{T}{\sqrt{f|_{r=r_h+h}}} \sim \frac{1}{l_p}, \quad (83)$$

which is also universal.

Actually, the local temperature cannot be much larger than the Planck scale in our analysis, otherwise the quantum correction will be needed.

The region of the momentum \vec{k} which gives the dominant contributions for this h is

$$|\vec{k}| \sim \frac{\sqrt{\rho}}{\sqrt{T h}} E \sim \frac{\sqrt{\rho}}{l_p} \sim \left(\frac{S_{BB}}{V} \right)^{1/d}, \quad (84)$$

which is the order of the entropy density.¹⁰

Finally, let us consider what happens if we assume the free energy of the background black brane and the one for the probe has different N scaling, as

$$N^\delta F_\phi \sim F_{BB} \sim N^\alpha. \quad (85)$$

For example, usual $N_c = N$ D3-branes with N_f flavor branes system give

$$\begin{aligned} F_{BB} &= N^2 T^{d+1} f_{BB}(g_{YM}, T), \\ F_\phi &= N_f N T^{d+1} f_\phi(g_{YM}, T), \end{aligned} \quad (86)$$

in our limit, where g_{YM} and T are fixed and N is very large and f_{BB} and f_ϕ are some functions. In this case, $\alpha = 2$, $\delta = 1$.¹¹ However in more generic D p -branes at non-'t Hooft limit and also branes in M-theory, α and δ vary, generically. We require that the free energy of the scalar field to be probe field yield,

$$0 \leq \delta \leq \alpha. \quad (87)$$

Below we will keep only l_p as a dimensional parameter. Since

¹⁰ Eqs. (83) and (84) imply that the quantum gravity effects cannot be neglected near $r = r_h + h$. The role of brick wall is to remove the degrees of freedom for the quantum gravity for local temperature/momentum above a cut off scale $\sim 1/l_p$. This can be done also by introducing the regulator which effectively cut off the geometry for $r \lesssim r_h + h$ without the explicit brick wall as [12].

¹¹ N D3-branes have N^2 degrees of freedom due to adjoint matrices, and N_f D7-branes have N degrees of freedom due to fundamental quarks. This is consistent with the famous 3/4 factor difference between $\lambda \rightarrow \infty$ limit and $\lambda \rightarrow 0$ limit [19,20].

$$F_{BB} \propto G_N^{-1} \propto l_p^{-d}, \quad F_\phi \propto h^{-\frac{d}{2}}, \quad (88)$$

from Eq. (79), this means

$$l_p \sim N^{-\frac{\alpha}{d}}, \quad \text{and} \quad h \sim N^{\frac{2(\delta-\alpha)}{d}} \sim (l_p)^{\frac{2(\alpha-\delta)}{\alpha}}. \quad (89)$$

Note that this l_p is defined from the relation $G_N \propto l_p^d$, so l_p is Planck length in the bulk AdS_{d+2} . Note also that the IR cut-off h in Eq. (89) is written *not* in terms of invariant distance but rather in terms of r . $h \sim l_p$ if $F_{BB} \sim (F_\phi)^2$.

Indeed, for the large black hole in the AdS coming from typical D3–D7 system in holographic QCD setting in the high temperature phase, the F_{BB} is expected to be scale like N^2 and the F_ϕ for the probe brane theory scale like N , since we have N D p -branes for $O(N^2)$ d.o.f. from adjoint representation and $O(1)$ probe D p' -branes for $O(N)$ d.o.f. from fundamental representation of $SU(N)$.

In the string theory context, usually there are Kaluza–Klein (KK) modes for the probe field. If the mass of a KK mode satisfies

$$m^2 \ll \frac{T}{h}, \quad (90)$$

then the KK modes give the same contribution to the free energy as the massless modes. The number of such modes will be $\left(\frac{T}{h}\right)^{\frac{\gamma}{2}}$, where γ is, typically, a dimension of the compactified manifold. Here the scale of the manifold was assumed to be $O(1)$. Thus including the KK modes, we have

$$F_\phi \propto h^{-\frac{d}{2}} \left(\frac{T}{h}\right)^{\frac{\gamma}{2}}, \quad (91)$$

instead of (88) and

$$h \sim (l_p)^{\frac{2(\alpha-\delta)}{\alpha} \frac{d}{d+\gamma}}. \quad (92)$$

Below we will assume that there are no KK-modes for notational simplicity because the essential discussions are not altered by replacing (89) with (92).

3.4. Charged scalar hair model

The effects of background charges for the black branes, which are dual to the chemical potential, are modification of the bulk momentum into covariant momentum. Therefore only modification due to the background flux is WKB approximated equation (21) should be replaced by

$$k_r(r) = \sqrt{g_{rr}(r) \left(-g^{tt}(r)(\mathcal{E})^2 - g^{ii}(r)(\tilde{k}_i)^2 - m^2 \right)} \quad (93)$$

where

$$\mathcal{E} \equiv E + qA_0, \quad \tilde{k}_i \equiv k_i - qA_i \quad (94)$$

The \vec{k} integration in $d\vec{k}k_r(r)$ is not modified since the integration range is still the range where the inside of the square-root in (93) is positive, and A_i gives just the integration parameter shift.

A_0 approaches zero at the horizon.¹² Therefore the effects of the flux do not modify essentially our the brick wall free energy divergence properties.

We have been considering minimal coupling. However as we have seen, the effect of the mass term is negligible for the brick wall argument, since $-E^2 g^{tt} \gg m^2$. Non-minimal couplings make the mass term position dependent, so this is effective mass changes. However as far as its value near the horizon does not diverge, still the mass term is negligible due to $-g^{tt} \rightarrow \infty$ near the horizon and our brick wall argument is not modified.

3.5. Probe brane model

We have seen the universal nature of the probe scalar fields. Note that we have never identified the bulk microscopic (or string theory) origin of the scalar fields. The scalar fields could be simple Kaluza–Klein mode from higher dimension compactified to lower dimensional AdS, or it could comes from some moduli fields in Calabi–Yau compactifications. It could be either closed string mode or open string mode. For example, in the D3–D7 system, open strings on the probe D7-brane, which are scalar “meson” fields, obey the similar wave equation as (2), from the probe DBI action in the $l_s \rightarrow 0$ limit.

Generically, if we consider introducing probe branes in the background, we will have some probe fields in the background at least in low energy limit. The contents of fields and the interactions are fixed by the type of probe branes and how they embed. However, on the black hole background geometry which has an infinite throat, the nature that low energy continuous spectrum appear for any fields on the probe branes are universal as far as probe brane are touching the black hole horizon. Thus, the above argument can be essentially applied for any probe fields.

Even for probe branes which does not cover the whole AdS space time, for example D8-branes in D4–D8, we expect that the branes will intersect the horizon of the black hole generically. In fact, for D8-brane in D4–D8 system see [27] for high finite temperature solutions where probe brane intersect with horizon.

Actually, if we consider larger and larger black holes in AdS, the branes eventually intersect the horizon and once horizon and probe brane intersect, essentially the same argument shows the probe meson free energy diverges.

3.6. Argument for fermions

We shall explicitly calculate minimally coupled Dirac fermion with the background gauge field. We work in the Lorentzian signature, while the Euclidean signature (which we employed in the derivation) can be easily obtained by an analytic continuation. First, we derive the Dirac equation. The fermion action in the bulk is

$$S_{\text{fermion}} = \int d^{3+1}x \sqrt{-g} i \left[\bar{\psi} \Gamma^M D_M \psi - m \bar{\psi} \psi \right]. \quad (95)$$

Here the Dirac operator is $D_M = \partial_M + \frac{1}{4} w_{abM} \Gamma^{ab} - iq A_M$. The definitions of the Gamma matrices in the local Lorentz frame are

¹² If A_0 does not vanish at the horizon, by analytically continuing it to the Euclid signature, it yields non-trivial Wilson loop $W = e^{i \int A_0 d\tau} = e^{i 2\pi A_0}$ on the tip of the (τ, r) cigar geometry. But since this is a contractible circle, it gives contradiction.

$$\Gamma^z \equiv \begin{pmatrix} \mathbf{1}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & -\mathbf{1}_2 \end{pmatrix}, \quad \Gamma^\mu \equiv \begin{pmatrix} \mathbf{0}_2 & \gamma^\mu \\ \gamma^\mu & \mathbf{0}_2 \end{pmatrix} \quad (96)$$

with $\gamma^0 \equiv i\sigma_3$, $\gamma^1 \equiv \sigma_1$, and $\gamma^2 \equiv -\sigma_2$, where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. We follow the notation of [30] and [17] except for the assignment of γ^μ (this difference is necessary to see the diagonalization as for fermion components, see below). The notation for the indices are: $M = 0, 1, 2, z$, and $\mu = 0, 1, 2$.

Writing the 4-component fermion as

$$\psi \equiv \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad \psi_\pm \equiv (-gg^{rr})^{-1/4} \phi_\pm, \quad (97)$$

where ϕ_\pm is a two-spinor, the Dirac equation is

$$\sqrt{\frac{g_{ii}}{g_{rr}}}(\partial_r \mp m\sqrt{g_{rr}})\phi_\pm = \mp iK_\mu \gamma^\mu \phi_\mp, \quad (98)$$

with $K_0 \equiv -i\sqrt{\frac{g_{ii}}{-g_{tt}}}(\partial_0 - iqA_0)$ and $K_i \equiv -i(\partial_i - iqA_i)$ with $i = 1, 2$.

The Dirac equation (98) is a coupled equation of ϕ_+ and ϕ_- , but one can eliminate one of them. Bringing (98) into the following form formally,

$$(-iK_\mu \gamma^\mu)^{-1} \sqrt{\frac{g_{ii}}{g_{rr}}}(\partial_r - m\sqrt{g_{rr}})\phi_+ = \phi_-, \quad (99)$$

$$(iK_\mu \gamma^\mu)^{-1} \sqrt{\frac{g_{ii}}{g_{rr}}}(\partial_r + m\sqrt{g_{rr}})\phi_- = \phi_+, \quad (100)$$

we can combine these to eliminate ϕ_- , to have

$$\left[(iK_\mu \gamma^\mu)^{-1} \sqrt{\frac{g_{ii}}{g_{rr}}}(\partial_r + m\sqrt{g_{rr}})(-iK_\mu \gamma^\mu)^{-1} \sqrt{\frac{g_{ii}}{g_{rr}}}(\partial_r - m\sqrt{g_{rr}}) \right] \phi_+ = \phi_+. \quad (101)$$

This is a second order differential equation for a two-spinor ϕ_+ , so we generically have four independent solutions. Without magnetic field, given E , four states are degenerate and they correspond to spin {up and down}, and {normalizable and non-normalizable} modes.

Under the WKB approximation, this equation becomes

$$\left[g_{ii} \left(g^{rr} \partial_r^2 - m^2 \right) \right] \phi_+ = (K_\mu \gamma^\mu)^2 \phi_+, \quad (102)$$

and with the ansatz $\phi_+ = e^{-iEt + i \int dr k_r(r) + i \tilde{k} \tilde{x}}$, we obtain

$$\begin{aligned} \left[g_{ii} \left(g^{rr} k_r^2 + m^2 \right) \right] &= -(K_\mu \gamma^\mu)^2 \\ &= - \left(-\sqrt{\frac{g_{ii}}{-g_{tt}}}(E + qA_0)\gamma^0 + (k_i - qA_i)\gamma^i \right)^2 \\ &\sim \frac{g_{ii}}{-g_{tt}}(E + qA_0)^2 - (k_i - qA_i)^2, \end{aligned} \quad (103)$$

so we obtain k_r as

$$k_r(r) = \sqrt{g_{rr}(r) \left(-g^{tt}(r)(\mathcal{E})^2 - g^{ii}(r)(\tilde{k}_i)^2 - m^2 \right)} \quad (104)$$

where

$$\mathcal{E} \equiv E + q A_0, \quad \tilde{k}_i \equiv k_i - q A_i. \quad (105)$$

Since fermions are self-conjugate, taking Dirichlet boundary condition for Dirac fermion at $r = r_h + h$ is too restrictive. Instead, we have to take an appropriate boundary condition such that only half of the degrees of freedom are fixed and also that energy flow at that boundary vanishes. By taking any such appropriate boundary conditions for the fermions, the same argument holds for fermions. Again, the difference from the background flux A_0 , A_i does not change the result. k_i integration is the same, A_i gives just a shift of the integration. Near the horizon, A_0 vanishes, therefore for the brick wall argument, this shows that essentially the same nature holds in fermion case with charged background.

4. Holographic interpretation of the brick wall

In the brick wall model, the Dirichlet boundary condition is imposed on the probe scalar field: $\phi(r = r_h + h) = C_{r_h+h}$. The scalar field is assumed to be free. The boundary is on $r = r_h + h$ with $h > 0$, i.e. outside the horizon. We first discuss various IR boundary conditions.

4.1. Various IR boundary conditions

In the Lorentzian AdS/CFT context, we usually take the ingoing boundary condition at the horizon since classically horizon keep absorbing objects. In fact, this choice of boundary condition allows us to evaluate the retarded Green function. However, it is also possible to take the Dirichlet boundary condition at least for probe fields once we go slightly away from the horizon.¹³ One can say that our boundary condition is on the “stretched horizon” [31]. Since it is a time-like surface, it is natural to consider either Dirichlet or Neumann boundary condition. Here we summarize some obvious properties of the Dirichlet (or Neumann) and ingoing boundary conditions.

Near the boundary at $r \approx r_h + h$, with $h > 0$, the equations of motion allow two independent approximated solutions with fixed E are $\phi \sim e^{i(Et+k_r(r_h)r)}$ and $\phi \sim e^{i(Et-k_r(r_h)r)}$, where $k_r(r_h)$ is the value of k_r in (21) evaluated at $r = r_h$, with $\text{Re}(E/k_r(r_h)) \geq 0$. From the action principle, we know not only the equation of motion but also what is a consistent boundary condition. If we take the ingoing boundary condition which means $\phi \sim e^{i(Et+k_r x)}$, it does not make the boundary term, $\phi \partial_r \phi$, vanish. Thus generically in such case the energy is not conserved through the boundary term and the amplitudes of scalar fields decay/amplify as time evolution. This is reflected to $\text{Im}(E) \neq 0$. Instead, if we take the Dirichlet boundary condition $\phi = C_{r_h+h}$ at $r = r_h + h$ with constant C_{r_h+h} , it makes the boundary condition satisfied only with $C_{r_h+h} = 0$. In this case, the eigenmodes are proportional to $e^{iEt} \sin(k_r(r_h)(r - r_h))$, where t is the static time outside the horizon, and we have real E , $\text{Im}(E) = 0$.

This can be seen explicitly from the bulk equation of motion as follows

$$0 = \int_{r_h+h}^{\infty} \sqrt{-g} \bar{\phi} \left(\frac{1}{\sqrt{-g}} \partial_r (\sqrt{-g} g^{rr} \partial_r \phi) E^2 g^{tt} \phi - k^2 g^{\bar{x}\bar{x}} \phi - m^2 \phi \right)$$

¹³ We should stress that we do not think these boundary conditions are really realized near the horizon. Instead, they are effective descriptions for quantum gravity, which represent where the semi-classical picture breaks down. As we will see later, this breakdown is related to the finite N effects. Thus, the usual computation without the brick wall (using the in-falling boundary condition) with the $1/N$ expansion may be consistent with the brick wall model.

$$\begin{aligned}
 &= \int_{r_h+h}^{\infty} \left(-(\partial_r \bar{\phi})(\sqrt{-g} g^{rr} \partial_r \phi) + \sqrt{-g} \left(-E^2 g^{tt} - k^2 g^{\bar{x}\bar{x}} - m^2 \right) \bar{\phi} \phi \right) \\
 &\quad + (\sqrt{-g} g^{rr} \bar{\phi} \partial_r \phi)_{r=r_h+h}.
 \end{aligned} \tag{106}$$

Here we consider the normalizable modes, i.e. $\phi \rightarrow 0$ as $r \rightarrow \infty$ so that the UV AdS boundary contribution vanishes, i.e., net energy flow at infinity.¹⁴ For $C_{r_h+h} = 0$, taking the imaginary part of this, we find $\text{Im}(E^2) = 0$ and $E^2 \geq 0$.¹⁵ This means that there are no tachyonic modes, even though there can be modes with $E^2 \ll \bar{k}^2$. For a general boundary condition, the imaginary part of this equality relation gives

$$\text{Im}(E^2) \int_{r_h+h}^{\infty} \sqrt{-g} (g^{tt} \bar{\phi} \phi) = (\sqrt{-g} g^{rr} \bar{\phi} \partial_r \phi)_{r=r_h+h}. \tag{107}$$

The right hand side surface term vanishes by $C_{r_h+h} = 0$ with Dirichlet boundary condition, or Neumann boundary condition, but not for Dirichlet boundary condition with $C_{r_h+h} \neq 0$ nor ingoing/outgoing boundary condition.

Remember that we need thermal equilibrium for the probe fields with the background black branes. This forces us that we cannot have a net energy flow between probe fields to the black branes therefore, only $C_{r_h+h} = 0$ with Dirichlet boundary condition, or Neumann boundary condition are the possible choices.

Note that in the original derivation of the probe field free energy in Section 2, as far as $g(E)$ in Eq. (16) is not qualitatively modified, we obtain the divergent free energy contributions. For $h \neq 0$, generic Dirichlet and Neumann boundary conditions are qualitatively the same in the sense that in both cases we obtain qualitatively the same $g(E)$. However ingoing/outgoing boundary conditions drastically change $g(E)$. Thus, the free energies with ingoing/outgoing or Dirichlet/Neumann boundary condition are completely different. Physically the spectrum obtained from the solutions with ingoing/outgoing boundary conditions has non-zero $\text{Im}(E) \neq 0$, so it is rather decaying object, due to the existence of the surface contribution.

Now we will consider the $h \rightarrow 0$ limit. For $h \rightarrow 0$, the solutions with the ingoing boundary condition are called the quasi-normal modes, [32,33]. The number of low energy quasi-normal modes is finite [32] and E has non-zero imaginary part. On the other hand, for the modes with the Dirichlet boundary condition with $h \rightarrow 0$, we cannot take $h \rightarrow 0$ limit as a well defined limit. To see this, let us study the near the horizon geometry, $r \sim r_h$, $ds^2 = -T(r - r_h)dt^2 + dr^2/T(r - r_h)$. By setting $\tilde{r} = dr/\sqrt{T(r - r_h)} = 2\sqrt{(r - r_h)/T}$, and $\theta = \pm i T t/2$, the metric approximates as

$$ds^2 \sim d\tilde{r}^2 + \tilde{r}^2 d\theta^2 + d\bar{x}^2 \tag{108}$$

which is a cigar geometry in Euclid signature. With the ansatz

$$\phi = \phi(r) e^{i l \theta}, \tag{109}$$

Laplacian gives

$$\square \phi = \frac{1}{\tilde{r}} \partial_{\tilde{r}} (\tilde{r} \partial_{\tilde{r}} \phi) - \frac{l^2}{\tilde{r}^2} \phi = 0 \tag{110}$$

¹⁴ We restrict the mass of the scalar field so that it has a normalizable and a non-normalizable modes for simplicity.

¹⁵ For $C_{r_h+h} \neq 0$, the charge density is not conserved at the boundary in general.

solution that allows power law behavior $\phi(\tilde{r}) \sim \tilde{r}^l$, $\phi \sim \tilde{r}^{-l}$. By setting $l = \mp 2iE/T$, $\theta = \pm iEt/2$, we have

$$\phi \sim \tilde{r}^{\mp 2iE/T} e^{iEt} \sim (r - r_h)^{\mp iE/T} e^{iEt}. \quad (111)$$

So the solutions are

$$\phi = (r - r_h)^{-iE/T} h_-(1 + \mathcal{O}(r - r_h)) + (r - r_h)^{iE/T} h_+(1 + \mathcal{O}(r - r_h)), \quad (112)$$

where terms proportional to h_- and h_+ represent the ingoing and outgoing modes, respectively. Imposing the Dirichlet boundary condition, we have

$$\frac{h_+}{h_-} = -e^{-2iE/T \ln h} (1 + \mathcal{O}(h)), \quad (113)$$

in which the phase oscillates infinitely. Thus in addition to the fact that the number of low energy modes diverges in $h \rightarrow 0$ limit, we cannot make the Dirichlet boundary condition well-defined in $h \rightarrow 0$ limit.

Let us discuss shortly the connection to the Euclidean case, since in that case, we have direct connection to the GKPW prescription [2,3] of the holographic Green function evaluation. For the $h \neq 0$ case, we have two solutions. So we can take any boundary conditions by combining these two solutions. On the other hand, at $h = 0$, one solution diverges so we *cannot* make any linear combination, in fact, it allows only *one* regular solution. This regular solution in Euclid signature goes to ingoing one in the Lorentzian signature as we have seen above Eq. (111). Actually by using this point and also the relationship between the Euclid Green function and retarded Green function

$$G_R(\omega, \vec{k}) = G_E(\omega_E, \vec{k})|_{\omega_E = -i(\omega + i\epsilon)}, \quad (114)$$

it was shown in [34] that one can obtain the retarded Green function, which was first proposed in [35] by bottom up approach, from the Euclid Green function using the GKPW prescription. For $h = 0$ case, since it is admitted only one solution from the regularity in Euclid signature, we do not have much freedom to obtain other Green function by using GKPW prescription. In that sense, $h \neq 0$ makes the situation better since it allows any linear combinations, as a results, any Green's functions.

4.2. Dual field theory interpretation of the divergence

Let us consider the background geometries dual to the confinement phase where the geometry has smooth IR “cut-off”¹⁶ at $r = r_c$ without black hole horizon. Examples for such geometries are global AdS which has a spherical boundary, and for R^4 boundary, we have Klebanov–Strassler [21] or D4-brane geometry compactified on S^1 [5]. These geometries are dual to the confinement phase in boundary theories and related to the scale $r = r_c$, they admit nonzero confinement/deconfinement transition at finite $T = T_c$. At temperature $T > T_c$, these geometries admit black hole horizon as [22]. If we add probe branes on these settings [23], then, in dual field theory, quark fields are introduced and we can study various meson confinement/deconfinement dynamics, see for example [24–27] and review [28]. The probe fields are quarks and whose degrees of

¹⁶ This “cut-off” is the bulk IR endpoint of the geometry related to the confinement scale in the boundary theory, not the IR cut-off of the brick wall at the deconfinement phase.

freedom are $\mathcal{O}(N)$ in the Lagrangian level.¹⁷ After a long time, the probe fields in the boundary theory will be thermalized with the gluons.¹⁸ In the bulk theory, this is interpreted as a thermalization of the probe fields in the bulk with a black hole.¹⁹ As we discussed in Section 3.5, we consider the brick wall model for the probe scalar fields on the probe branes under such thermal equilibrium.

For $T < T_c$, they are confined into the mesons. The number of *low energy* spectrum of the mesons is finite, namely there is a mass gap and the spectrum is discrete and low energy free energy does not diverge. This is consistent with the spectrum of the scalar in the bulk Lorentzian (global) AdS_d analysis.

For $T > T_c$, the quarks will be deconfined. Therefore, in the gravity dual we expect that there are $\mathcal{O}(N)$ contributions, which become divergent free energy in the large N limit. This divergence is exactly what we have seen in the bulk analysis of Sections 2 and 3 in the $h \rightarrow 0$ limit, namely without the brick wall cut-off. Note that if we take the ingoing boundary condition, then we do not have such behavior at least. Since it diverges, in the large but finite N case, we need to put regulator near the horizon $r = r_h + h$, with $h \neq 0$. For a boundary condition there, we can take Dirichlet or Neumann boundary condition for example. This helps, since as we have seen in Section 2.4, the divergence comes from the divergence of the number of the *low energy* degree of freedom, namely $E \rightarrow 0$ modes. This E corresponds to the energy conjugate to the boundary time, due to our choice of the ansatz (7).²⁰ Since the Dirichlet boundary condition suppresses the low energy mode divergence and changes the spectrum from continuous to discrete, in field theory dual, this corresponds to introduction of the *IR cut-off* by the finite N effect. This is the brick wall in a holographic context. Note that we expect that at infinite N with nonzero coupling λ , the spectrums of the gauge theories at the deconfinement phase are continuous as is seen in the analysis [36–39]. On the other hand, finite N makes the spectrums discrete even at the deconfinement phase. The essential role of finite N is to introduce the IR cut-off and remove the divergent IR degeneracy by making the continuous spectrum into discrete spectrum at the deconfinement phase, and as a result, the free energy becomes finite. The brick wall with nonzero h plays the role of the finite N effect in the dual field theory, since finiteness of N makes the spectrum discrete in the boundary theory at the deconfinement phase, and finiteness of $1/h$ makes the spectrum discrete in the bulk theory under the presence of a black hole.

The free energy will increase and diverge if we take $h \rightarrow 0$. However we know that the probe D7-branes introduce at most $\mathcal{O}(N)$ (quarks) contributions in the Lagrangian level, and it cannot exceed the $\mathcal{O}(N^2)$ contribution of the background D3 degrees of freedom. This gives the lower bound for h , where at least, naive picture allowed by local quantum field theory in curved space–time is not trustworthy. As we discussed in the end of Section 3.3, this implies that the trust in local field theory for counting bulk d.o.f. under the given metric (1) will be lost at $h \sim l_p$, i.e. $r = r_h + \mathcal{O}(l_p)$ at least, for the black hole in the AdS space–time. More precisely, the semi-classical free field approximation for counting d.o.f. on the geometry will not be valid.

In more generic setting,

¹⁷ We neglect the effect of the massive open string modes by the non-’t Hooft limit which we take, $l_s \sim l_p \ll 1$. These massive open strings are dual to massive vector mesons such as ρ mesons.

¹⁸ The backreaction of the probe fields to the gluons will be small in the large N limit.

¹⁹ The backreactions to the black hole geometry by the probe fields will be small because the black hole mass is large.

²⁰ The continuity of the low energy spectrum for the free energy divergence can be seen in other places. For example, by considering the scalar fields in Poincaré coordinate pure AdS space, where we have zero temperature horizon, we can see that the spectrum becomes continuous and as a result, their free energy diverge, see for example [28].

$$N^\delta F_\phi \sim F_{BB} \sim N^\alpha, \quad (115)$$

we have seen that the IR cut-off h is determined as

$$h \sim (l_p)^{\frac{2(\alpha-\delta)}{\alpha}} \quad (116)$$

in Eq. (89). Without h , the free energy of the probe field F_ϕ diverges. Since we expect that the metric and semi-classical description for counting d.o.f. are trustworthy from far infinity up to $O(1)$ ²¹ close to the horizon $r \gtrsim r_h + O(1) \gg r_h + h$, this implies that h must be at most positive power of Planck scale, especially, it cannot be independent of the Planck scale.²² Therefore, it is reasonable to expect $\alpha > \delta$, so

$$F_\phi \sim N^{\alpha-\delta} \rightarrow \infty \quad (\text{in the large } N \text{ limit}) \quad (117)$$

Therefore, F_ϕ always diverges by the positive powers of N .

Of course, these arguments have several loopholes. One should note that we have *assumed* that at finite l_p , quantum gravity effects are not important at $r \gtrsim r_h + O(1) \gg r_h + h$, therefore we have squeezed the quantum gravity effects at the region $r \lesssim r_h + h$. However we do not have good argument to justify this assumption. It is interesting to study this more concretely in holographic setting.

Note that for the argument (117), we have assumed that the boundary theory does not have momentum space cut-off satisfying (38). For D3–D7 [23] or D4–D8 [26] system, the theory in UV approaches the scale invariant theories and they do not admit any cut-off Λ . However if there is momentum space cut-off satisfying (38), then the conclusion (117) breaks down and the cut-off h could be bigger.

Note also that the brick wall model requires the thermal equilibrium between the black hole and the probe field, and this is possible with the large black hole in the AdS space–time, since it allows positive specific heat. Moreover, the thermal equilibrium implies that there is no net energy flow between the probe fields and the black holes. In dual field theory, this is between probe quarks and the background gluons. Zero value Dirichlet boundary condition and Neumann boundary condition make the surface term vanish, but ingoing and outgoing boundary conditions allow the nonzero surface term and therefore there are energy flows, so they will not allow a thermal equilibrium.

The divergence of the free energy F_ϕ we have evaluated is, in fact, a one-loop effect and the reader might wonder at the tree level contribution. In fact, as is seen in [40,41] the classical Dirac–Born–Infeld (DBI) action evaluated on the Euclidean black hole background yields the $O(\lambda N_f N)$ results. Note that this is the same as Gibbons–Hawking prescription [42], and $O(\lambda N_f N)$ contribution is due to the horizon existence. This is correct in the $N \rightarrow \infty$ limit with λ fixed. On the other hand, in the brick wall context where N is large but rather finite, we do not expect the sharp notion of the horizon existence; full quantum gravity effects become important and geometry is not trustworthy slightly outside of the horizon, which is described by the brick wall at $r = r_h + h$. So tree level contributions are actually expected to be $O(1)$ [43] in

²¹ $O(1)$ implies AdS scale or any other scales, which are independent of Planck or string scale.

²² Note that we are considering the limit where g_s is fixed finite as $g_s \sim 1$ and $N \rightarrow \infty$ so that string scale and Planck scale are the same order and both are very small, $l_s \sim l_p \rightarrow 0$, compared with AdS curvature scale or large black hole curvature scale. Therefore in our limit, there is no distinction between Planck corrections and stringy corrections since both appear simultaneously in our limit.

our case. Therefore this one-loop divergence can be dominant.²³ We emphasize that as we have interpreted, this IR divergence is physical in the sense that it is associated with the continuity of the spectrum in the boundary theory. We will discuss more in the discussion section for the finite N effect as an IR cut-off.

4.3. Difficulty of having a fundamental “electron” in the dual field theories

We have seen that under the black hole background, there is universal nature for the probe free energy F_ϕ ; it always diverges due to the continuous gapless spectrum. The universality of the brick wall implies that all the probe fields show divergent free energy by the same mechanism. What does this mean in a boundary context? We interpret this as a signal that all the probe fields which are singlet at the confinement phase, show the continuous gapless spectrum with large IR degeneracy at the deconfinement phase. So they are actually not the fundamental fields such as electrons but rather they are at most composite fields.

What this brick wall model argument is suggesting is that the holographic dual field theory can have gauge-singlet fields ($SU(N)$ singlet) only as composite fields, like mesons.²⁴ In other words, the brick wall model suggests that the gauge singlet components should always show the confinement–deconfinement transition and their degrees of freedom (in the Lagrangian level) should scale by a positive power of N , therefore, we can have electron like objects as composite objects like mesons, or fermionic superpartners of mesons to make them fermionic.

To see this, let’s consider adding an $SU(N)$ gauge-singlet fundamental field (in the Lagrangian level), like an electron field, to the $SU(N)$ QCD like theory. Suppose that the gauge-singlet fundamental field becomes thermal equilibrium, then we can consider its free energy. The contribution to the free energy from the gauge singlet field should be independent of N , so it is always $O(1)$ even after the $SU(N)$ QCD like theory shows the confinement–deconfinement transition. However this contradicts the gravity analysis; if we consider the free energy of the bulk scalar field, which is dual to the gauge singlet field, it shows the divergence without the brick wall cut-off due to the near horizon throats in the vicinity of the black holes with $h = 0$. With IR cut-off $h \neq 0$, free energy behaves as $\sim 1/h$. Note that this divergence is due to the almost continuity of the spectrum in the IR, therefore its degrees of freedom cannot be $O(1)$. Therefore, they cannot be gauge singlet fields after the confinement–deconfinement transition, but rather they are composite fields at most.²⁵

The fact that they always show these divergences is due to the universality of the gravity. Gravity couples to all the fields, furthermore gravitational theory, which has asymptotically AdS boundary, always forms a large black hole at high temperature. Therefore, as we have seen in Sections 2 and 3, any bulk fields dual to the gauge-singlet matter fields obey their equations of motion, and their free energy always shows the divergence due to the localized modes near the black hole horizon, at least in large N limit.

²³ Usually, the 1-loop effects give $O(1)$ contributions. However, in our case the number of the low energy modes is infinite and it is $O(1) \times \infty$ contribution in the $N \rightarrow \infty$ limit, i.e. $h \rightarrow 0$ limit. Then, because N is finite and h is finite, the 1-loop effects can give $O(N)$. On the other hand, in the usual $1/N$ expansion picture, tree level contribution is $O(N)$.

²⁴ This is the argument for theories which have some UV cut-off energy scale and we assume that this UV cut-off is much bigger than the confinement–deconfinement transition.

²⁵ One possible loophole of this argument is that this discrepancy is due to the strong coupling λ effect, since gravity is at large λ but gauge theory is at small λ , so free energy behaves as a positive power of λ , instead of N . However in the limit we are considering, where $g_{YM} = O(1)$, we do not distinguish the large λ effect and large N effect since $\lambda \sim N$.

How much the cut-off is necessary in the gravity side is evaluated from the field theory sides. If we add the glueballs, then their free energy and entropy scale as $O(N^2)$ at the deconfinement phase. This makes the free energy of the dual bulk scalar fields the same order as background black brane free energy (which is dual to the gluons), and this fix the h to be Planck scale in the invariant distance. On the other hand, if we introduce the probe brane as we discussed in the previous subsection, then free energy of the meson scalar is $O(N)$ at the deconfinement phase, which is suppressed by N from the black brane entropy. This yields the new IR cut-off scale. Again, note that the IR cut-off h in Eq. (116) is *not* invariant distance but rather distance in terms of r -coordinate.

4.4. Path-integral measure from Euclidean analysis

So far we have seen the divergence from the Lorentzian signature. Since after the analytic continuation, the bulk geometry near the black branes or holes becomes simply smooth cigar geometry where no metric components show coordinate singularities, unlike Lorentzian signature. Therefore one might wonder in Euclid signature, if these divergences disappear. However since we have seen the divergences are physical from dual field theory viewpoint, they should not disappear even after the analytic continuation. Here we will conduct the partition function evaluation by Euclidean path-integral method and see that the divergences still exist. This subsection is mainly a review.

For the partition function evaluation in Euclidean path-integral analysis, we can consider the usual Matsubara formalism. We will see that the path-integral measure for this is different from the diffeomorphism invariant one with the conformal transformation by g_{tt} [44–47].

By interpreting partition function as Euclid time evolution by β ,

$$Z = \text{Tr} e^{-\beta H} = \int [d\phi] e^{-\int d\tau L_E(\phi)} \quad (118)$$

We will evaluate this path-integral in this subsection.

For that purpose, let us recall the analysis in Lorentzian signature. We will define the complete set motivated by the Lorentzian signature Laplacian ∇_L^2 ,

$$\nabla_L^2 = g^{tt} \partial_t^2 + \nabla_{r,x}^2 = g^{tt} \left(\partial_t^2 + g_{tt} \nabla_{r,x}^2 \right) \quad (119)$$

such that complete sets we define are

$$g_{tt}(r) \nabla_{r,x}^2 \phi_{n,r,x} = E_n^2 \phi_{n,r,x}, \quad (120)$$

with the normalization

$$-\int dr dx \sqrt{-g} g^{tt} \phi_{n,r,x} \phi_{m,r,x} = \delta_{n,m}. \quad (121)$$

Note that this gives precisely Eq. (8). Then we expand the field as

$$\phi(t, r, x) = \sum_n c_n(t) \phi_{n,r,x}. \quad (122)$$

The solutions of the equation of motion are given by $c_n(t) = c_n e^{iE_n t}$. Then the action is

$$\int dt L = \int dt \sum_n \left(\dot{c}_n(t)^2 - E_n^2 c_n(t)^2 \right), \quad (123)$$

the harmonic oscillators for quantum mechanical modes $c_n(t)$. Thus, the Hamiltonian is

$$H = \sum_{n=0}^{\infty} (m_n + \frac{1}{2}) E_n \quad (124)$$

for the eigenstates,

$$| \{m_n\} \rangle = \prod_{n=0}^{\infty} (a_n^\dagger)^{m_n} | 0 \rangle \quad (125)$$

labeled by $\{m_n\}$ where $m_n = 0, 1, 2, \dots$, and this reduces to the Section 2 analysis.

Motivated by this Lorentzian analysis, we will evaluate Eq. (118). In the path-integral conducted in Euclid signature, we will expand the field as

$$\phi(\tau, r, x) = \sum_n c_n(\tau) \phi_{n,r,x}, \quad (126)$$

where we choose the complete sets $\phi_{n,r,x}$ similar to the one defined in Eq. (120) in Lorentzian case, i.e., we choose,

$$-g_{\tau\tau}(r) \nabla_{r,x}^2 \phi_{n,r,x} = E_n^2 \phi_{n,r,x}. \quad (127)$$

The Euclid Lagrangian L_E gives

$$\begin{aligned} & \int d\tau L_E(\phi) \\ &= \int d\tau dr d\vec{x} \sqrt{g} g^{\tau\tau} \sum_{m,n} \left(\dot{c}_n(\tau) \dot{c}_m(\tau) \phi_{n,r,x} \phi_{m,r,x} + (E_m)^2 c_n(\tau) c_m(\tau) \phi_{n,r,x} \phi_{m,r,x} \right) \\ &= \int d\tau \sum_n \left(\dot{c}_n(\tau)^2 + E_n^2 c_n(\tau)^2 \right) \\ &= \sum_n \sum_m \left(\left(\frac{2\pi m}{\beta} \right)^2 + E_n^2 \right) (c_{nm})^2 \end{aligned} \quad (128)$$

where we have used the periodicity under the $\tau \rightarrow \tau + \beta$ from the trace property to expand $c_n(\tau) \sim \sum_{m \in \mathbb{Z}} c_{nm} e^{\frac{2\pi i}{\beta} m \tau}$ and the completeness condition

$$\int dr dx \sqrt{g} g^{\tau\tau} \phi_{n,r,x} \phi_{m,r,x} = \delta_{n,m} \quad (129)$$

as a normalization.

This yields the path-integral evaluation of the partition function (118) up to a numerical normalization constant,

$$Z = \Pi_n \Pi_{m \in \mathbb{Z}} \left(\left(\frac{2\pi m}{\beta} \right)^2 + E_n^2 \right)^{-\frac{1}{2}}. \quad (130)$$

Consider the background which is pure AdS. Then the partition function determined by this prescription gives the thermodynamical partition function of the, for example, mesons. This justifies that prescription above gives the correct answer from the AdS/CFT viewpoint.

On the other hand, the above prescription gives the divergent contribution on the path-integral method in view of Euclid prescription. This divergence is due to

$$g^{\tau\tau} = \infty, \quad (131)$$

for the complete sets (127), since then, we have large n mode with fixed energy and this yields the free energy divergence. The large n mode has bigger value for the r derivatives. The existence of huge n with fixed $E \sim T$ is the origin of the divergence.

Naively one might think that because in Euclid signature, the black hole is replaced by the cigar geometry so there is no warped factor, at least locally. However, since the normalization must be chosen in such a way (127), the effect of (131) still induces the divergence for the partition function.

If we use the different complete sets, for example,

$$-\nabla_{r,x}^2 \phi_{n,r,x} = E_n^2 \phi_{n,r,x} \quad (132)$$

this does not diagonalize the path-integral, therefore it does not reduce to the simple determinant, unlike (128).

On the geometry far outside the horizon, we believe that usual quantum field theory in curved space-time should hold as a good approximation. This means that we should path-integrate the scalar field on the geometry as usual. However as we have seen, the treatment for the basis choice in Eqs. (127) and (129) treats diffeomorphism in a non-invariant way, since we treat $g^{\tau\tau}$ specially and this induces divergence as Eq. (131). The reason for treating $g^{\tau\tau}$ special must be clear. In order to define energy or temperature for thermal partition functions, we have to specify their conjugate, which is time, that is why we treat metric component $g^{\tau\tau}$ specially.

5. Discussion

Probe fields in the bulk, which are dual to the probe fields like meson fields, show the divergent free energy due to the near horizon throat of black holes. The origin of the free energy divergence is due to the large degeneracy of the modes, living near the throat region of the horizon $r - r_h \gtrsim h$, with

$$E \lesssim \sqrt{\frac{T}{h}}, \quad \vec{k}_{local}^2 \lesssim \frac{T}{h} \quad (133)$$

from the bulk theory viewpoint. We have seen that the value of cut-off h depends on the microscopic theories. In the microscopic system where the degrees of freedom for background black brane F_{BB} and probe fields F_ϕ are given by

$$F_{BB} \sim N^\alpha, \quad F_\phi \sim N^{\alpha-\delta}, \quad (134)$$

the appropriate IR cut-off h is

$$h \sim (l_p)^{\frac{2(\alpha-\delta)}{\alpha}} \quad (135)$$

otherwise, semi-classical approximation over-counts the degrees of freedom as we have seen in Section 4.2. For example, in the Dp – Dp' brane setting with $(p, p') = (3, 7)$, where we have N_c Dp branes with probe N_f Dp' branes, we have $\alpha = 2$ and $\delta = 1$. These probe Dp' -branes introduce microscopic quark fields which have no mass gap and continuous spectrum in the boundary theory at the deconfinement phase. In this way, we have seen that these divergences are not pathological, but rather physical consequence, at least for the probe brane models from the holographic viewpoint. Note that from the bulk viewpoint, if we do not remove the modes

$$E \gg \sqrt{\frac{T}{h}}, \quad \vec{k}_{local}^2 \gg \frac{T}{h}, \quad (136)$$

then the free energy of the probe field becomes bigger than the microscopic counting (134), which is inconsistent. Taking only the modes satisfying (133) but not the modes satisfying (136), we obtain a large but non-diverging free energy as (134).

In Section 2.4 we have seen that after the k integration, we have enhancement for $1/h$ by the contributes $g_{rr} \sim 1/h$ factor. This implies that large $|k|^2 \sim T/h$ does not give large E due to the near horizon warping effect. In dual field theory, this is interpreted as due to the interaction with the background gluon plasmas, which are dual to black holes.

In the dual field theory, the semi-classical limit corresponds to the large N limit and then the free energy for the deconfinement phase quarks should scale as $\mathcal{O}(N)$, which is divergent. From this microscopic analysis, it should be answered what are the cut-off effects in the gravity side, which corresponds to the finite N effects. We have seen that once we specify the microscopic degrees of freedom for both probe fields and background black branes, then the cut-off is obtained as appropriate positive power of the Planck scale.

What we have seen is that the semi-classical path-integral method for counting the degrees of freedom diverges in the large N limit from both bulk and boundary viewpoint. But on large but finite N , bulk semi-classical path-integral method over-counts due to the modes (136). Therefore, this implies that the semi-classical approximation and our trust for the naive picture of the black hole geometries might break down for $r - r_H \lesssim h$. The point that the semi-classical approximation breaks down near the horizon is indeed what we expect from the recent firewall argument or fuzzball argument, [48–51], although, these arguments are based on other viewpoints: the inconsistency between unitarity of black hole evaporations and local quantum field theory in curved space near the horizon. Our argument is different; it is rather based on the over-counting issues of the degrees of freedom in the semi-classical limit, and also on how the finite N or l_p effects should appear in the bulk.

In full quantum gravity, the geometric picture probably breaks down. However, it might be reasonable to *assume* that at least the geometry is very well approximated to the black hole geometry as far as $r \gg r_H$. Therefore, it might be reasonable to expect that the probe field free energy divergence is cured and the modes (136) are effectively removed by the quantum gravity effects at $r \lesssim r_H + h$.

The role of the finite N effect in the boundary theory reduces the degrees of freedom, compared with $N \rightarrow \infty$ limit. From a boundary theory viewpoint, these finite N regulators effectively reduce the ∞ by ∞ matrices into finite N by N .

Note that we expect that at infinite N with nonzero coupling λ , the spectrums of the gauge theories at the deconfinement phase are continuous as is seen in the analysis [36–39]. On the other hand, finite N makes the spectrums discrete even at the deconfinement phase. Therefore, the reduction of degrees of freedom by finite N makes the spectrum from continuous to discrete even at the deconfinement phase, and this discreteness is crucial to see the Poincaré recurrence of the time-evolution of the unitarity. Apparent non-unitarity of the black hole information loss is due to the continuity of the spectrum at the deconfinement phase. With infinite N the phase space volume is expected to be infinite²⁶ while with finite N , the phase space volume is expected to be finite, and no physical quantities should diverge. If the spectrum is continuous, the Green functions do not show Poincaré recurrence and decay in late time as [36–39]. In that aspect, our argument is also implicitly relying on the unitarity nature of the quantum mechanical system.

²⁶ Note that if the spectrum is discrete even at $N \rightarrow \infty$ limit as is $\lambda \rightarrow 0$ case, then the phase space volume becomes finite and the time-scale of the Poincaré recurrence is finite, contradicted with the classical information loss picture.

From a bulk theory, introducing both a non-zero cut-off h and a Pauli–Villars regulator like [12] has an effect to reduce the UV degrees of freedom above Planck scale (measured in the local Lorentz frame) as Eq. (136) with Eq. (44). Therefore, by introducing the brick walls, the spectrum also changes from the continuous one to discrete one. The point that these effects are UV in bulk theory but are IR in boundary theory is due to the UV/IR mixing in holography [52, 53]. Therefore, finite N effects are IR cut-off in a boundary theory to make the spectrum discrete, and in the bulk, this is what brick wall does with $h \neq 0$. Brick wall effects with finite $1/h$ are the same as finite N effects in the boundary theory and these are IR cut-off effects for the boundary theory spectrum.

The reductions of degrees of freedom from classical limit due to the non-perturbative quantum gravity effects are expected to be generic. In fact, they are seen using the gauge/gravity correspondence for many situations. Finite N matrix model for M -theory [1, 54–56] yields the quantum gravity reduction. In [57], the reduction by finite N occurs due to the “droplet” quantization, also in [39], the discreteness nature of the Young tableaux reduces the degrees of freedom. Such reductions of degrees of freedom in full quantum gravity are important and the cut-off h of the brick wall model reflects this reduction. This suggestion might be important since this gives another estimate of the where the locality breaks down, and deviation from classical gravity limit occurs.

Originally, the brick wall was introduced for regularization of the divergence of the free energy. This divergence can be absorbed into the renormalized Newton constant [12, 31]. If our goal is simply to obtain finite numbers for the free energy of the probe fields, the classical DBI action evaluation near the horizon gives the correct non-diverging free energy in the large N leading order [40, 41], and the brick wall may not be needed. From holographic boundary theory viewpoint, this Newton constant renormalization can be regarded as an effective N change as

$$F_{BB}(N^2) + F_\phi(N) = F_{BB}(N_{eff}^2), \quad (137)$$

so that $O(N)$ contribution of F_ϕ is absorbed as an effective N change into N_{eff} for F_{BB} . This might be meaningful if N is infinite and one can be regarded $1/N$ as an almost continuous parameter. However in finite N , it is unclear if such an effective N change makes sense. Similarly, the problem behind the divergence is the *almost* continuous low energy spectrum in large N . Thus, the real advantage of the brick wall picture is for a finite N , i.e. a finite l_p theory, where the naive classical picture of horizons does not work. It is interesting to investigate more on the difference between the large N picture and the finite N picture, where non-perturbative quantum gravity effects are dominating.

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